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A Transfer Theorem

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1. Introduction

In this paper, we will prove a transfer theorem which generalizes the P. Hall–Wielandt theorem ([2], Theorem 14.4.2).

Let G be a finite group. Suppose that a subgroup H of G contains the normalizer of an S_p-subgroup of G for some prime p. Then, the image of G by the transfer V of G into $H/H'(p)$ is a direct factor of $H/H'(p)$ ([5] and [11]). Here, $H'(p)$ denotes the smallest normal subgroup of H with an abelian factor group of order a power of p. We will prove the following theorem.

THEOREM. *We have the direct product decomposition*

$$H/H'(p) = V(G) \times (L/H'(p))$$

where L is generated by $H'(p)$ and the elements of H which are fused to the commutators $[x, y; p-1]$ $(x, y \in P)$.

The notation is standard:

$$[x, y; 0] = x, \qquad [x, y; i+1] = [[x, y; i], y],$$

so $[x, y; p-1] = [x, y, \ldots, y]$ is the higher commutator in which y appears $p-1$ times.

If the S_p-subgroup P contains a subgroup Q which is weakly closed in P with respect to G and $H \supseteq N_G(Q)$, then we can only use the commutators

$$[x, y; p-1]$$

with $x \in P$ and $y \in Q$ in the above theorem.

In the particular case when $H'(p) \supseteq L$, the theorem asserts that

$$H'(p) = H \cap G'(p) \qquad \text{and} \qquad G/G'(p) \cong H/H'(p).$$

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This implies the isomorphism

$$G/O^p(G) \cong H/O^p(H)$$

by a theorem of Tate [8]. Thus, we obtain the P. Hall–Wielandt theorem.

In a recent paper, Yoshida [11] reexamined the theory of transfer from the view-point of the representation theory, and showed, in particular, how the subgroup theorem of Mackey is used to obtain various results about transfer. The approach of this paper is essentially the same as Yoshida's, but we will follow the idea of Isaacs [5] in viewing the subgroup theorem as a means of computing the transfer. Thus, we will use only the customary concept of transfer, not the character theoretic transfer of [11].

The second section will review the works of Yoshida and Isaacs. The transfer theorem will be proved next. The last section will contain an application of this method to a simple group which contains a large extra-special 2-subgroup.

2. Preliminaries

Let K be a normal subgroup of H with an abelian factor group H/K. The transfer V of G into H/K is defined as follows:

Choose a transversal T of H in G; the set T is, by definition, a collection of representatives of the cosets of form tH. If x is an element of G, for each $t \in T$, there are two elements

$$t(x) \in T \qquad \text{and} \qquad h(x, t) \in H$$

such that the equality

$$xt = t(x)\, h(x, t)$$

holds. The transfer V is a mapping defined on G into H/K by

$$V(x) = \prod h(x, t)K,$$

where t ranges over all elements of T. The transfer has the following important property.

(2.1) The value $V(x)$ of the transfer is independent of the transversal T, and V is a homomorphism of G into H/K.

If, in particular, the normal subgroup K is the commutator subgroup H' of H, then the transfer of G into H/H' is simply called the transfer of G into H. Any transfer of G into H/K is just a composite of the transfer of G into H and the canonical map of H/H' onto H/K.

Let L be a subgroup of G which contains H. Then, the transitivity of transfer may be expressed as follows.

(2.2) Let W be the transfer of L into H/K and let U be the transfer of G into L. If x is an element of G, then we have

$$V(x) = W(y)$$

where y is any element of L such that $U(x) = yL'$.

Symbolically, we have

$$V(x) \equiv W(U(x)).$$

This proposition depends on (2.1) and the fact that the commutator subgroup L' is contained in the kernel of W. Thus, (2.2) is true for any transfer U_0 of G into L/L_0, provided that

$$L_0 \subseteq \text{Ker } W$$

and that y satisfies the equation $U_0(x) = yL_0$.

The following proposition corresponds to the subgroup theorem of Mackey (see [11] and [5]).

(2.3) Let H and P be subgroups of G and let

$$G = Ps_1H \cup Ps_2H \cup \cdots \cup Ps_mH$$

be the decomposition of G into disjoint union of double cosets with respect to P and H. Set $P_i = P \cap s_iHs_i^{-1}$ for each i. Let V be the transfer of G into a factor group H/K and let V_i denote the transfer of P into P_i. Then, for an element x of P, we have

$$V(x) = \prod s_i^{-1}v_i(x)\, s_iK$$

where $v_i(x)$ is an element of P_i such that

$$V_i(x) = v_i(x)P_i'.$$

As in the proof of the subgroup theorem, this is proved because the set

$$T_1s_1 \cup T_2s_2 \cup \cdots \cup T_ms_m$$

is a transversal of H in G, where T_i is a transversal of P_i in P.

Proposition (2.3) is useful in stating the relationship between the transfer V of G and the local transfers V_i of P. If the subgroup P is, in particular, the cyclic subgroup generated by x, then $v_i(x)$ is the first power of x which lies in P_i. Thus, (2.3) gives the usual formula for computing transfer $V(x)$.

The following definition corresponds to Definition 3.4 of [11].

DEFINITION 2.4. Let p be a fixed prime. A quadruplet

$$(G, H, K, x)$$

consisting of a finite group G, two subgroups H and K of G, and an element x of G is called *singular* if the following conditions are satisfied:

(1) x is an p-element of G;

(2) $K \lhd H$ and the factor group H/K is the cyclic group of order p;

(3) $V(x) \neq 1$ for the transfer V of G into H/K;

(4) no element of $\langle x^p \rangle$ fuses to an element of $H - K$.

We will denote the set of all the singular quadruplets by $\mathscr{S}$.

If a group G contains a normal subgroup N of index p, then

$$(G, G, N, x) \in \mathscr{S}$$

for a p-element x of minimal order in $G - N$. However, for a p-group G and its proper subgroup H, the condition

$$(G, H, K, x) \in \mathscr{S}$$

is very restrictive, as shown by the following lemma from [11].

(2.5) Let P be a p-group. Suppose that for a proper subgroup H of P we have

$$(P, H, K, x) \in \mathscr{S}.$$

Let L be a subgroup of P such that $P \supseteq L \supseteq H$. We denote the transfer of L into H/K by W and set $N = \operatorname{Ker} W$. Then, the following propositions hold:

(a) $(P, L, N, x) \in \mathscr{S}$;

(b) there is an element y of L such that y fuses to x and

$$(L, H, K, y) \in \mathscr{S};$$

(c) if L is a maximal subgroup of P, then $x \in L$;

(d) if L is a maximal subgroup of P, then

$$P/\operatorname{Core}_P N \simeq Z_p \operatorname{wr} Z_p ,$$

where $\operatorname{Core}_P N$ is the largest normal subgroup of P which is contained in N, and the right side denotes the standard wreath product of two cyclic groups of order p.

Using the usual formula for $V(x)$, (a) can be proved by a routine check of the condition (4) of Definition 2.4 and the application of Proposition (2.2). Similarly, (b) is proved. To prove (c) and (d), suppose that L is maximal. Then $L \lhd P$. If z is an element of $P - L$, the transfer U of P into L/N satisfies the formulas

$$U(z) = z^p N \qquad \text{and} \qquad U(y) = \prod z^{-i} y z^i N$$

for any element y of L. This, together with (a), yields (c) and (d).

Statements (a) and (b) hold even without the assumption on $|P|$. They correspond to Lemma 3.3, (3) and (4) of [11] respectively. Statement (d) is clearly a sufficient condition for

$$(P, L, N, x) \in \mathcal{S}$$

with $x \in L - N$, and is also Lemma 3.6(4) of [11]. The property (c) is implicit in [11], but remarked in [5]. Together with (b), when applied to a subgroup L with $[L : H] = p$, it implies that the element x fuses in H.

The following is Lemma 2.4(1) of [11] (see [5]).

(2.6) Suppose that the index of a subgroup H is not divisible by the prime p. Then, we have

$$H/H'(p) = V(G) \times L/H'(p)$$

where V is the transfer of G into $H/H'(p)$ and

$$L = H \cap G'(p).$$

3. The Transfer Theorem

In this section we consider an S_p-subgroup S of a group G and a subgroup Q of S which is weakly closed in S with respect to G. Thus, for $x \in G$, $Q^x \subseteq S$ implies that $Q^x = Q$. Let Q_0 be the subgroup of S which is generated by the elements fused to higher commutators

$$[x, y; p - 1] \qquad (x \in S, y \in Q).$$

We will prove the following theorem.

Theorem 1. *Let H be a subgroup of G which contains $N_G(Q)$. Then, the following direct product decomposition holds*:

$$H/H'(p) = V(G) \times (Q_0 H'(p)/H'(p))$$

where V is the transfer of G into $H/H'(p)$.

This theorem is equivalent to any one of these statements:

$$H \cap G'(p) = Q_0 H'(p);$$
$$S \cap G'(p) = Q_0(S \cap H'(p));$$
$$\mathrm{Foc}_G(S) = Q_0 \,\mathrm{Foc}_H(S),$$

where $\mathrm{Foc}_G(S)$ denotes the focal subgroup with respect to G.

The equivalence of the first two is clear. The last one is just a rewriting of the second by the focal subgroup theorem [4].

Suppose that the theorem fails. Then, $Q_0H'(p)$ is a proper subgroup of $H \cap G'(p)$, so there is a normal subgroup K of index p in H such that $K/H'(p)$ contains both $V(G)$ and $Q_0H'(p)/H'(p)$.

Choose an element x of minimal order in $S - K$. We set $P = \langle Q, x \rangle$ and apply (2.3).

The transfer W of G into H/K is a composite of the transfer V and the canonical map of $H/H'(p)$ onto H/K. Since $K/H'(p)$ contains the image $V(G)$, we have $W(x) = 1$. Thus, by using (2.3), we obtain

$$1 = W(x) = \prod s_i^{-1} v_i(x)\, s_i K,$$

and $W_i(x) = v_i(x)P_i'$, where $P_i = P \cap s_i H s_i^{-1}$ and W_i is the transfer of P into P_i. If $s_i = 1$, then $P_i = P$ and $v_i(x) = x$. Since the element x is not contained in K, there must be a double coset Ps_iH such that

$$Ps_iH \neq H \qquad \text{and} \qquad s_i^{-1} v_i(x)\, s_i \notin K.$$

We set $K_i = P \cap s_i K s_i^{-1}$. Then, we check the containment

$$(P, P_i, K_i, x) \in \mathscr{S}.$$

It is important that P_i is a proper subgroup of P. If $P_i = P$, then

$$s_i^{-1} P s_i \subseteq H.$$

By the definition of P, $s_i^{-1} Q s_i$ is contained in H. Since Q is weakly closed in S and $N_G(Q) \subseteq H$ by definitions, the element s_i lies in H. This contradicts the relation $Ps_iH \neq H$.

Let L be a maximal subgroup of P which contains P_i. By (2.5c) the element x lies in L, so by the definition of P, we have $P = QL$. We can choose an element y of $Q - L$, and set

$$z = [x, y; p - 1].$$

In the notation of (2.5), set $N_0 = \mathrm{Core}_P\, N$. Then, for the transfer U of P into L/N_0, we have

$$U(x) = zN_0,$$

because L/N_0 is isomorphic to $Z_p \operatorname{wr} Z_p$ by (2.5d). Since N_0 is contained in the kernel of the transfer of L into P_i/K_i, (2.2) is applicable and yields

$$V_i(x) = U_0(z)$$

where V_i is the transfer of P into P_i/K_i and U_0 is the transfer of L into P_i/K_i. The usual formula for $U_0(z)$ shows that $U_0(z)$ is a coset containing the product of elements $t^{-1}z^n t$ of P_i. But

$$t^{-1} z^n t \in s_i H s_i^{-1},$$

so $s_i^{-1}t^{-1}z^n t s_i$ is conjugate to an element of Q_0 in H. Since $Q_0 \subseteq K \lhd H$, we have $t^{-1}z^n t \in s_i K s_i^{-1}$. This implies that $U_0(z) = 1$, contrary to $V_i(x) \neq 1$. The theorem is proved.

The proof actually shows that $H/H'(p)$ is generated by the image $V(G)$ of transfer and $Q_0H'(p)/H'(p)$. Clearly we have

$$Q_0H'(p) \subseteq H \cap G'(p).$$

If the index of H in G is prime to p, then

$$G = HG'(p),$$

so we have

$$[H : H \cap G'(p)] = [G : G'(p)] \leqslant |V(G)|.$$

This yields that $Q_0H'(p) = H \cap G'(p)$ and that

$$H/H'(p) = V(G) \times (Q_0H'(p)/H'(p))$$

without using (2.6).

4. Remarks

An extra-special 2-group E is called a *large* extra-special subgroup of G if the following two conditions are satisfied:

$$E = O_2(C_G(Z(E))) \qquad \text{and} \qquad C_G(E) \subseteq E.$$

Many sporadic simple groups and Chevalley groups over the field of two elements have an involution whose centralizer possesses a large extra-special 2-subgroup. Simple groups with large extra-special 2-subgroups were studied recently by Aschbacher [1], Harada [3], Smith [6, 7], and Timmesfeld [9].

In this section we will prove the following theorem using an elementary transfer method.

Theorem 2. *Let E be a large extra-special 2-subgroup of order >32 in a group satisfying $G = O^2(G)$. Set $H = C_G(Z(E))$. If K is a normal subgroup of index 2 in H, then there is an involution of E which fuses to an element of $H - K$.*

As a corollary we get the result of Harada [3].

Corollary. *Under the assumptions of Theorem 2, the factor group $H/H'(2)$ is elementary.*

We state several properties of an extra-special 2-group which are required in the proof.

(4.1) Let E be an extra-special 2-group, let M be a maximal subgroup of E, and let L be a maximal subgroup of M. Suppose that $|E| \geqslant 32$.

(a) E has no homomorphism onto the dihedral group of order 8.

(b) If L is abelian, the order of E is 32.

(c) $E - M$ contains an involution.

(d) If $|E| > 32$, $M - L$ contains an involution as well as an element of order 4.

(e) If an involutive automorphism σ satisfies $C_E(\sigma) = M$, then σ is an inner automorphism of E.

The following proposition will be proved.

(4.2) Let E be a large extra-special 2-subgroup of G. Set $H = N_G(E)$. Suppose that $|E| > 32$ and that

$$G = O^2(G) \qquad \text{but} \quad H \neq O^2(H).$$

Let K be a subgroup of index 2 in H. Then, the following assertions hold:

(i) $E \subseteq K$;

(ii) there is an involution of E which fuses in $H - K$;

(iii) the generator z of $Z(E)$ fuses in $E - \{z\}$.

The first statement (i) is a theorem of Smith [6]. Aschbacher [1] and Smith [7] have classified all simple groups which do not satisfy (iii). Thus, the results (i) and (iii) are known, but our method yields them without much extra work. (For (i), the proof works even if $|E| = 32$.)

We choose a 2-element x in $H - K$ as follows:

Case 1. If $E \nsubseteq K$, then x is an involution of $E - K$;

Case 2. If $E \subseteq K$, choose x as a 2-element of minimal order in $H - K$.

By (4.1c), $E - K$ contains an involution, so this choice of x is possible in both cases. Set

$$P = \langle E, x\rangle.$$

Since E is large, E is weakly closed in H with respect to G. Thus, a double coset PsH contains more than one coset of H unless $PsH = H$.

As in the proof of the previous theorem, (2.3) yields the existence of a double coset $Ps_iH \neq s_iH$ such that

$$v_i(x) \in P_i - K_i \qquad (K_i = P \cap s_iKs_i^{-1}).$$

Thus, P_i is a proper subgroup of P and

$$(P, P_i, K_i, x) \in \mathscr{S}.$$

By (2.5d), P has a homomorphism onto $Z_2 \operatorname{wr} Z_2$, the dihedral group of order 8. So, (4.1a) yields that $P \neq E$. The definition of the element x shows that case 1 does not occur; thus, we have (i).

Let T be a maximal subgroup of P which contains P_i. By (2.5c) we have $x \in T$, so $P = ET = EP_i$. Set $M = E \cap T$. Then, M is a maximal subgroup of E. For any element u of $E - M$, the commutator $[x, u]$ is an element of M, but does not belong to the kernel of the transfer W of T into P_i/K_i. By (4.1c), M contains an element w of order 4 such that

$$W(w) \neq 1.$$

Since $P = EP_i$, we have $T = MP_i$. Set

$$J = M \cap P_i \qquad \text{and} \qquad L = M \cap K_i.$$

In computing the transfer $W(w)$, we can choose a transversal which consists of elements of M. Hence if U denotes the transfer of M into J/L, we have

$$U(w) \neq 1.$$

This shows that the quadruplet (M, J, L, w) satisfies the first three conditions of Definition 2.4.

Suppose that $(M, J, L, w) \notin \mathscr{S}$. Then $w^2 = z$ fuses in $J - L$. Thus, the element z fuses in $H - K$, so (ii) holds.

If (iii) fails, then w^2 fuses to no element of $J - L$ except z. In this case $J = L \times \langle z \rangle$ is an elementary abelian subgroup, so no element of J is conjugate to w. Thus,

$$U(w) = z^n L, \qquad \text{where} \quad [M : J] = m = 2n.$$

As $U(w) \neq 1$, we have $n = 1$; J is then a maximal subgroup of M. By (4.1b) we conclude $|E| = 32$.

Finally, assume that $(M, J, L, w) \in \mathscr{S}$. In this case the element w fuses in J. Since any cyclic subgroup of order 4 of E is normal, w is an element of J. Then J contains $z = w^2$, so $J \lhd M$. We have $z \in L$ and

$$U(w) = w^m L, \qquad \text{where} \quad m = [M : J].$$

Hence $m = 1$ and $M = J$. The elements of $J - L$ fuse in $H - K$, so (4.1d) yields the assertion (ii).

The equality $M = J$ implies that $M \subseteq s_i H s_i^{-1}$. Set $z^* = s_i z s_i^{-1}$. Then we have $z^* \neq z$. By definition $s_i H s_i^{-1} = C_G(z^*)$, so z^* centralizes M. In particular

$[z^*, z] = 1$ and $z^* \in H$. The element z^* induces an involutive automorphism σ which satisfies the assumption of (4.1e). Hence there is an element u of E such that $uz^* \in C_G(E)$. This yields $z^* \in E$, since $C_G(E) \subseteq E$ by hypothesis.

We will add a few remarks on a singular quadruplet

$$(G, H, K, x).$$

If P is a p-subgroup of G which contains x, then there is an element s of G such that

$$(P, P \cap sHs^{-1}, P \cap sKs^{-1}, x) \in \mathscr{S}.$$

This follows from (2.3), but $P \cap sHs^{-1}$ may (and often does) coincide with P. If P is a p-group and if

$$(P, Q, R, x) \in \mathscr{S},$$

then $N_P(\langle x \rangle)$ is conjugate to a subgroup of Q ([11], Lemma 3.8(4)). This indicates that Q is fairly large. In fact, we have

$$| Q | \geqslant p^2[P : Q]^{p-2}.$$

If $p > 2$, the equality holds only for $P = Z_p \operatorname{wr} Z_p$. Also, when $p > 2$, $| Q | \geqslant p^2[P : Q]$ (cf. [11], Lemma 3.8(5) and Lemma 3.9(5)). On the other hand, for $p = 2$, $| Q | = 4$ is possible. If $P = \langle x, y \rangle$ is a dihedral group such that

$$x^2 = y^{2n} = 1 \qquad \text{and} \qquad x^{-1}yx = y^{-1},$$

then for $Q = \langle x, y^n \rangle$ and $R = \langle x \rangle$, we have

$$(P, Q, R, x) \in \mathscr{S}.$$

This shows another disparity of 2 and the other primes.

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